Finite Element Analysis of Transient Dynamic Viscoelastic Problems in Time Domain

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In this paper, the simplified and stable finite element method is presented for the time domain analysis of the transient dynamic viscoelastic problems, for which the weak form is obtained by applying the Galerkin's method to the equations of motion in time integral which do not contain the inertia terms explicitly, but the inertia effect is taken into account, and discretized spatially to obtain the semidiscrete equations in time integral. In the temporal approximation, only the time interpolation functions are used for approximating the dependent variables on the divided time axis, while the time integration schemes such as the Newmark and Houbolt methods are not necessary in contrary to the conventional approach. To show the validity and applicability, two-dimensional examples are given and solved for the displacements and stresses, especially for the dynamic stress concentrations by the wave diffraction, which are discussed in detail at the aspect of the viscoelastic damping. To the authors' knowledge, no previous results except for the test example exist in the literature.

Key Words : Viscoelastic, Dynamic Stress Concentration, Wave Propagation, Finite Element

1. Introduction

The equations of motion in terms of displacements in the linear dynamic viscoelasticity are of integrodifferential and differential forms with respect to the time and space variables, respectively, and so the solving procedures become more complicated compared to those of the quasi-static viscoelasticity and the dynamic elasticity, the stable and accurate numerical methodologies being required.

Up to now, the transient dynamic linear viscoelastic problems have been analyzed in the time and transformed (Laplace or Fourier) domains mainly using the numerical tools such as the FEM (Finite Element Method) (Barrett and Gotts, 2002 ; Golla and Hughes, 1985 ; Goudreau, 1970 ; Ha et al., 2002 ; Liu and Sharan, 1995 ; Nickell, 1968, 1971 ; Spyrakos, 1987 ; Yi and Hilton, 1994), the BEM (Boundary Element Method) (Gaul and Schanz, 1999 ; Manolis and Beskos, 1981 ; Pérez-Gavilán and Aliabadi, 2001 ; Polyzos et al., 1994), and the FDM (Finite Difference Method) (Beskos and Leung, 1984 ; Chen and Cheng, 2000 ; Dey and Rao, 1997 ; Li et al., 1992), which are sometimes combined with the sophisticated methods like the fractional derivative model (Bagley and Torvik, 1985 ; Eldred et al., 1996 ; Enelund et al., 1999) and the spectral element technique (Doyle, 1988 ; Lee and Kim, 2001).

By the way, the equations of motion in integrodifferential form in the transient dynamic viscoelasticity can be transformed into the equations of motion in time integral through the use of convolution or the Laplace transform and its inversion, and which concept has been partially used in the numerical analysis of the transient

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dynamic elasticity (Sim and Lee, 2002). But the applications of latter equations to the transient dynamic viscoelastic analysis are very few. Formerly, Nickell (1968 ; 1971) and Goudreau (1970) derived the semidiscrete finite element equations by taking the first variation of the spatially discretized Leitman's variational functional (Oden and Reddy, 1976). In the temporal approximation, Goudreau introduced a one step quadrature formula similar to the Newmark method, and Nickell expressed the displacement as a quadratic function of time to insure the continuity of the displacement and velocity between the time intervals on the discretized time axis, but it was found later that his method produces a negative damping (i.e., unconditionally unstable). So, Nickell modified his method to be unconditionally stable using the technique similar to the Wilson's averaging method. But their methods still requires the time-integration schemes for the velocity and acceleration and their applications have been limited only to one-dimensional problems.

In this paper, the simplified and stable finite element equations in matrix form for the time domain analysis of the transient dynamic viscoelastic problems are newly presented based on the equations of motion in time integral, for which the weak form is obtained by applying the Galerkin's method to those equations and discretized spatially to obtain the semidiscrete equations in time integral. In the temporal approximation, the time integration schemes such as the Newmark and Houbolt methods (Bathe, 1996) are not necessary since the inertia terms are disappeared in those equations. Instead, only the time interpolation functions are used to approximate the dependent variables on the discretized time axis, resulting in an implicit time integration scheme. The viscoelasticity matrix is derived by applying the elastic-viscoelastic correspondence principle to the elasticity matrix for the viscoelastic material which behaves elastically in dilatation and like a standard linear solid in shear.

To show the validity and applicability of the presented method, two-dimensional examples with infinite and finite domains are solved for the displacements and stresses, especially for the dynamic stress concentrations by the wave diffraction, to the authors' knowledge, which solutions except for the test example are given for the first time in this paper, and the influences of the viscoelastic damping on the wave propagation are discussed in detail.

2. Weak Formulation

The governing equations of the linear dynamic viscoelasticity (Christensen, 1982) are similar to those of the linear dynamic elasticity (Achenbach, 1975) except the stress-strain relations of hereditary integral type and can be written as follows :

(i) Equations of motion

$$\sigma_{ij,j}(\boldsymbol{x},\ t)+\rho f_i(\boldsymbol{x},\ t)=\rho\ddot{u}_i(\boldsymbol{x},\ t) \tag{1}$$

where σ_{ij} is the stress, ρ the mass density, f_i the body force per unit mass, u_i the displacement, $\boldsymbol{x}$ the position vector, and t the time variable.

(ii) Strain-displacement relations

$$\varepsilon_{ij}(\boldsymbol{x},\ t)=\frac{1}{2}\{u_{i,j}(\boldsymbol{x},\ t)+u_{j,i}(\boldsymbol{x},\ t)\} \tag{2}$$

where ε_{ij} is the small strain tensor.

(iii) Stress-strain relations

$$\begin{aligned}\sigma_{ij}(\boldsymbol{x},\ t)&=D_{ijkl}(t)*d\varepsilon_{kl}(\boldsymbol{x},\ t)\\&=\int_{0^-}^{t}D_{ijkl}(t-\tau)\frac{\partial\varepsilon_{kl}(\boldsymbol{x},\ \tau)}{\partial\tau}\,d\tau \qquad (3a)\\&=\varepsilon_{kl}(\boldsymbol{x},\ t)*dD_{ijkl}(t) \qquad (3b)\end{aligned}$$

where $D_{ijkl}(t)$ is the viscoelasticity matrix of relaxation type, the operator $*$ means the Stieltjes convolution as defined in Eq. (3a), and the viscoelastic material is assumed to be undisturbed before the external force is applied at $t=0$.
And the boundary and initial conditions are given by

$$\begin{aligned}&u_i(\boldsymbol{x},\ t)=\hat{u}_i(t)\ \text{on}\ \Gamma_u,\ T_i(\boldsymbol{x},\ t)=\hat{T}_i(t)\ \text{on}\ \Gamma_t\\&u_i(\boldsymbol{x},\ 0)=d_i(\boldsymbol{x}),\ \dot{u}_i(\boldsymbol{x},\ 0)=v_i(\boldsymbol{x})\ \text{at}\ t=0\end{aligned} \tag{4}$$

where t_i is the traction, Γ_u and Γ_t are the portions of the boundary $(\Gamma=\Gamma_u+\Gamma_t)$ where the displacement and traction are specified, and d_i and v_i are the prescribed initial values for the displacement and velocity, respectively.

If Eq. (2) is substituted into Eq. (3a) and then the expression for the stresses is subsequently substituted into Eq. (1), the equations of motion in terms of displacements are obtained as

$$G(t)*du_{i,jj}(x,\ t)+\{\lambda(t)+G(t)\}*du_{k,ki}(x,\ t) +\rho f_i(x,\ t)=\rho\ddot{u}_i(x,\ t) \tag{5}$$

where $G(t)$ and $\lambda(t)$ are the relaxation moduli corresponding to the Lamé constants in the isotropic linear elasticity. Eq. (5) is the integro-differential equations of motion with respect to the time.

Through the use of convolution or the Laplace transform and its inversion, Eq. (1) can be transformed into the equations of motion in time integral in terms of stresses.

$$g*\sigma_{ij,j}+g*\rho f_i-\rho(-tv_i-d_i+u_i)=0 \tag{6}$$

where $g=g(t)=t$ and the convolution in Eq. (6) is defined as $g(t)*f(t)=\int_0^t g(t-\tau)f(\tau)\,d\tau$.

Eq. (6) contains the initial conditions implicitly and no inertia terms, which is equivalent to Eq. (1) and known as the Euler equations of the variational functional for the linear dynamic viscoelasticity by Leitman (Oden and Reddy, 1976).

In this paper, the weak form for the time-domain finite element analysis of the transient dynamic linear viscoelastic problems is obtained by applying the Galerkin's method to Eq. (6). That is,

$$\int_\Omega [g*\sigma_{ij,j}+\rho\{g*f_i+(tv_i+d_i)\}-\rho u_i]\,\delta u_i d\Omega=0 \tag{7}$$

where Ω represents the spatial domain.

By applying the Gauss' theorem and Cauchy's stress formula to Eq. (7) and employing the relations $\sigma_{ij}\delta u_{i,j}=\sigma_{ij}\delta\varepsilon_{ij}$ and arranging, the weak form is obtained under the assumption of no body forces as

$$\int_\Omega g*\sigma_{ij}\delta\varepsilon_{ij}d\Omega+\int_\Omega \rho u_i\delta u_i d\Omega =\int_\Gamma g*t_i\delta u_i d\Gamma+\int_\Omega \rho(tv_i+d_i)\,\delta u_i d\Omega \tag{8}$$

Note that the inertia terms are disappeared in Eq. (8) so that the time-integration schemes such as the Newmark and Houbolt methods (Bathe, 1996) to approximate the acceleration and velocity are not necessary in this work.

3. Finite Element Equations

For the development of Eq. (8), the time axis is divided equally and then the dependent variables are approximated on the divided time interval. At this time, the constant time variation is adopted because it has brought on unconditionally stable numerical results in the elastodynamic wave propagation analysis (Sim and Lee, 2002). Then the displacements are approximated by the linear combinations of spatial and time functions as

$$u_i(x,\ t)=\sum_{n=1}^{N}\Phi_n(t)\,u_i^n(x) \qquad 0<t\le t_N \tag{9}$$

where $\Phi_n(t)$ are the global time interpolation functions on the discretized time axis and $\Phi_n(t)=1$ on $t_{n-1}\le t\le t_n$ and $\Phi_n(t)=0$ otherwise. The arbitrary and current time nodes are expressed by $t_n=n\Delta t$ and $t_N=N\Delta t$, respectively, and $u_i^n(x)$ is the spatial distribution of the displacements in a time interval $t_{n-1}\le t\le t_n$.

By using Eq. (3a) and the commutativity law of convolution, the first term on the left-hand side of Eq. (8) can be written as

$$\int_\Omega g*\sigma_{ij}\delta\varepsilon_{ij}d\Omega=\int_\Omega E_{ijkl}*d\varepsilon_{kl}\delta\varepsilon_{ij}d\Omega \tag{10}$$

where,

$$E_{ijkl}(t)=g(t)*D_{ijkl}(t) \tag{11}$$

In Eq. (10), The elements of $E_{ijkl}(t)$ consist of t and t^2 functions in addition to the exponential functions which are the elements of the stiffness matrix of the finite element equations for the quasi-static viscoelasticity. $D_{ijkl}(t)$ and $E_{ijkl}(t)$ will be derived in the next section. By the integration by parts, the convolution on the right-hand side of Eq. (10) can be written as

$$E_{ijkl}(t)*d\varepsilon_{kl}(x,\ t)=E_{ijkl}(0)\,\varepsilon_{kl}(x,\ t) -\int_{0^+}^{t}\varepsilon_{kl}(x,\ \tau)\frac{dE_{ijkl}(t-\tau)}{d\tau}d\tau \tag{12}$$

It is assumed that the strains vary stepwise on the discretized time axis as in Eq. (9). That is,

$$\varepsilon_{ij}(\boldsymbol{x}, t)=\sum_{n=1}^{N}\Phi_n(t)\,\varepsilon_{ij}^n(\boldsymbol{x}) \tag{13}$$

where $\varepsilon_{ij}^n(\boldsymbol{x})$ is the spatial distribution of the strains in the nth time interval.

Substitution of Eq. (13) into the second term on the right-hand side of Eq. (12) yields

$$\begin{aligned}&-\int_{0^+}^{t}\varepsilon_{kl}(\tau)\frac{dE_{ijkl}(t-\tau)}{d\tau}\,d\tau\\&=-\int_{0^+}^{t}\sum_{n=1}^{N}\Phi_n(\tau)\,\varepsilon_{kl}^n(\boldsymbol{x})\frac{dE_{ijkl}(t-\tau)}{d\tau}\,d\tau\\&=-\sum_{n=1}^{N}C_{ijkl}(n\Delta t)\,\varepsilon_{kl}^n(\boldsymbol{x})\end{aligned} \tag{14}$$

where,

$$\begin{aligned}C_{ijkl}(n\Delta t)=&E_{ijkl}\{(N-n)\Delta t\}\\&-E_{ijkl}\{(N-(n-1))\Delta t\}\end{aligned} \tag{15}$$

Substituting Eq. (14) into Eq. (12) and arranging, we get

$$\begin{aligned}E_{ijkl}(t)*d\varepsilon_{kl}(\boldsymbol{x}, t)=&-\sum_{n=1}^{N-1}C_{ijkl}(n\Delta t)\,\varepsilon_{kl}^n(\boldsymbol{x})\\&+E_{ijkl}(\Delta t)\,\varepsilon_{kl}^N(\boldsymbol{x})\end{aligned} \tag{16}$$

where the relations $\varepsilon_{kl}(\boldsymbol{x}, t)=\varepsilon_{kl}^N(\boldsymbol{x})$ are used. Substituting of Eq. (16) into Eq. (10), and the resulting equation into Eq. (8), we obtain

$$\begin{aligned}&\int_\Omega E_{ijkl}(\Delta t)\,\varepsilon_{kl}^N(\boldsymbol{x})\,\delta\varepsilon_{ij}d\Omega+\int_\Omega\rho u_i\delta u_i d\Omega\\&=\int_\Gamma g*t_i\delta u_i d\Gamma+\int_\Omega\rho(tv_i+d_i)\,\delta u_i d\Omega\\&\quad+\sum_{n=1}^{N-1}\int_\Omega C_{ijkl}(n\Delta t)\,\varepsilon_{kl}^n(\boldsymbol{x})\,\delta\varepsilon_{ij}d\Omega\end{aligned} \tag{17}$$

The first term on the right hand side of Eq. (17) can be calculated analytically if the external forces are given, and so, for example, when the external force is applied suddenly at $t=0$ and then keep constant thereafter, the results are obtained as

$$\int_\Gamma g*t_i\delta u_i d\Gamma=\frac{t^2}{2}\int_\Gamma\hat{t}_i(\boldsymbol{x})\,\delta u_i d\Gamma \tag{18}$$

where $t_i(\boldsymbol{x}, t)=\hat{t}_i(\boldsymbol{x})H(t)$, and $H(t)$ is the Heaviside step function.

Substituting Eq. (18) into Eq. (17) and writing in matrix form, the finite element equations for the analysis of the transient dynamic linear viscoelastic problems are obtained as follows:

$$\begin{aligned}&\int_\Omega[B]^T[E(\Delta t)][B]d\Omega\{u_i\}^N\\&+\int_\Omega\rho[N]^T[N]d\Omega\{u_i\}^N\\&=\frac{t^2}{2}\int_\Gamma[N]^T\{\hat{t}_i(x)\}d\Gamma\\&+\int_\Omega[N]^T\rho(tv_i+d_i)\,d\Omega\\&+\sum_{n=1}^{N-1}\int_\Omega[B]^T[C(n\Delta t)][B]d\Omega\{u_i\}^n\end{aligned} \tag{19}$$

In the derivation of Eq. (19), the following relations are used.

$$\begin{aligned}\{\delta u_i(\boldsymbol{x}, t)\}&=[N]^T\{\delta u_i\}^N\\\{\delta\varepsilon_{ij}(\boldsymbol{x}, t)\}&=[B]^T\{\delta u_i\}^N\end{aligned} \tag{20}$$

where, $\{\delta u_i\}^N$ is the virtual displacement vector at $t=t_N$, $[N]$ is the shape function matrix, $[B]$ is the strain-displacement matrix, and the matrices $[C(n\Delta t)]=[C_{ijkl}(n\Delta t)]$ and $[E(\Delta t)]=[E_{ijkl}(\Delta t)]$ will be derived hereafter.

From Eq. (11),

$$\begin{aligned}E_{ijkl}(t)&=\int_0^t g(t-\tau)*D_{ijkl}(\tau)\,d\tau\\&=-t\bar{D}_{ijkl}(0)+\bar{\bar{D}}_{ijkl}(t)-\bar{\bar{D}}_{ijkl}(0)\end{aligned} \tag{21}$$

where,

$$\begin{aligned}\bar{D}_{ijkl}(t)&=\int_0^t D_{ijkl}(\tau)\,d\tau\\\bar{\bar{D}}_{ijkl}(t)&=\int_0^t\bar{D}_{ijkl}(\tau)\,d\tau\end{aligned} \tag{22}$$

4. Stress Computation in Matrix Form

In the numerical analysis of the transient dynamic linear viscoelastic problems, the viscoelastic material is assumed to behave elastically in dilatation and like a standard linear solid in shear as shown in Fig. 1, which are expressed by

$$G(t)=G_0\{\alpha+(1-\alpha)e^{-\lambda t}\},\ K(t)=K_0 \tag{23}$$

where $G(t)$ and $K(t)$ are the relaxation moduli in shear and dilatation, respectively, and G_0 is the initial shear relaxation modulus, K_0 is the elastic bulk modulus, α is the ratio between the asymptotic and initial relaxation moduli ($0<\alpha<1$), and λ is the inverse of the relaxation time t_R.

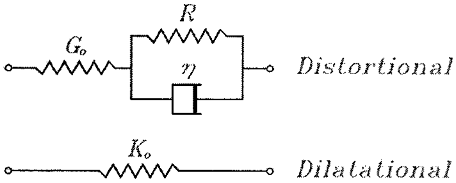

Fig. 1 Viscoelastic material

4.1 Derivation of $E_{ijkl}(t)$

The viscoelasticity matrix $D_{ijkl}(t)$ or $[D(t)]$ for an isotropic linear viscoelastic material can be derived by applying the elastic-viscoelastic correspondence principle to the elasticity matrix $[D]$ (Zienkiewicz and Taylor, 1991) for an isotropic linear elastic material and is given as follows for the case of plane stress.

$$[D(t)]=\begin{bmatrix} d_{11}(t) & d_{12}(t) & 0 \\ d_{21}(t) & d_{22}(t) & 0 \\ 0 & 0 & d_{33}(t) \end{bmatrix} \tag{24}$$

where,

$$d_{11}(t)=d_{22}(t)=p_1+p_2e^{-\gamma_1 t}+p_3e^{-\gamma_2 t}$$

$$d_{12}(t)=d_{21}(t)=q_1+q_2e^{-\gamma_1 t}+q_3e^{-\gamma_2 t}$$

$$d_{33}(t)=w_1+w_2e^{-\lambda t}$$

and the coefficients p_1, p_2, p_3, q_1, q_2, q_3, w_1, w_2, γ_1 and γ_2 are listed in the Appendix.

Matrices $[\bar{D}(t)]$ and $[\bar{\bar{D}}(t)]$ are obtained by integrating the viscoelasticity matrix $[D(t)]$ once and twice with respect to the time, respectively, and the latter is given as

$$[\bar{\bar{D}}(t)]=\begin{bmatrix} \bar{\bar{d}}_{11}(t) & \bar{\bar{d}}_{12}(t) & 0 \\ \bar{\bar{d}}_{21}(t) & \bar{\bar{d}}_{22}(t) & 0 \\ 0 & 0 & \bar{\bar{d}}_{33}(t) \end{bmatrix} \tag{25}$$

where,

$$\bar{\bar{d}}_{11}(t)=\bar{\bar{d}}_{22}(t)=-\left(\frac{p_2}{\gamma_1^2}+\frac{p_3}{\gamma_2^2}\right)+\left(\frac{p_2}{\gamma_1}+\frac{p_3}{\gamma_2}\right)t+\frac{p_1}{2}t^2+\left(\frac{p_2}{\gamma_1^2}e^{-\gamma_1 t}+\frac{p_3}{\gamma_2^2}e^{-\gamma_2 t}\right)$$

$$\bar{\bar{d}}_{12}(t)=\bar{\bar{d}}_{21}(t)=-\left(\frac{q_2}{\gamma_1^2}+\frac{q_3}{\gamma_2^2}\right)+\left(\frac{q_2}{\gamma_1}+\frac{q_3}{\gamma_2}\right)t+\frac{q_1}{2}t^2+\left(\frac{q_2}{\gamma_1^2}e^{-\gamma_1 t}+\frac{q_3}{\gamma_2^2}e^{-\gamma_2 t}\right)$$

$$\bar{\bar{d}}_{33}(t)=-\frac{w_2}{\lambda^2}+\frac{w_2}{\lambda}t+\frac{w_1}{2}t^2+\frac{w_2}{\lambda^2}e^{-\lambda t}$$

And we get

$$[\bar{D}(0)]=[\bar{\bar{D}}(0)]=0 \tag{26}$$

Substituting Eq. (26) into Eq. (21), the following relations are obtained.

$$E_{ijkl}(t)=\bar{\bar{D}}_{ijkl}(t) \tag{27}$$

Using the relations of Eq. (27), matrix $C_{ijkl}(n\Delta t)$ or $[C(n\Delta t)]$ in Eq. (15) can be written as

$$[C(n\Delta t)]=\begin{bmatrix} c_{11}(n\Delta t) & c_{12}(n\Delta t) & 0 \\ c_{21}(n\Delta t) & c_{22}(n\Delta t) & 0 \\ 0 & 0 & c_{33}(n\Delta t) \end{bmatrix} \tag{28}$$

where,

$$c_{11}(n\Delta t)=c_{22}(n\Delta t)=-\left(\frac{p_2}{\gamma_1}+\frac{p_3}{\gamma_2}\right)\Delta t-\frac{p_1}{2}(2N-2n+1)(\Delta t)^2+\frac{p_2}{\gamma_1}e^{-\gamma_1(N-n)\Delta t}(1-e^{-\gamma_1\Delta t})+\frac{p_3}{\gamma_2^2}e^{-\gamma_2(N-n)\Delta t}(1-e^{-\gamma_2\Delta t})$$

$$c_{12}(n\Delta t)=c_{21}(n\Delta t)=-\left(\frac{q_2}{\gamma_1}+\frac{q_3}{\gamma_2}\right)\Delta t-\frac{q_1}{2}(2N-2n+1)(\Delta t)^2+\frac{q_2}{\gamma_1^2}e^{-\gamma_1(N-n)\Delta t}(1-e^{-\gamma_1\Delta t})+\frac{q_3}{\gamma_2^2}e^{-\gamma_2(N-n)\Delta t}(1-e^{-\gamma_2\Delta t})$$

$$c_{33}(n\Delta t)=-\frac{w_2}{\lambda}\Delta t-\frac{w_1}{2}(2N-2n+1)(\Delta t)^2+\frac{w_2}{\lambda^2}e^{-\lambda(N-n)\Delta t}(1-e^{-\lambda\Delta t})$$

In the case of plane strain, the elements of the viscoelasticity matrix $[D(t)]$ are defined as in Eq. (29), and the coefficients p_4, p_5, p_6, q_4, q_5, q_6, w_1, w_2, ξ_1 and ξ_2 are listed in the Appendix.

$$d_{11}(t)=d_{22}(t)=p_4+p_5e^{-\xi_1 t}+p_6e^{-\xi_2 t}$$

$$d_{12}(t)=d_{21}(t)=q_4+q_5e^{-\xi_1 t}+q_6e^{-\xi_2 t} \tag{29}$$

$$d_{33}(t)=w_1+w_2e^{-\lambda t}$$

4.2 Stress computation

Stresses can be computed using either Eq. (3a) or (3b), but the latter is adopted in this work and rewritten in matrix form as

$$\begin{aligned}\{\sigma(\boldsymbol{x},t)\}^T&=\int_{0^-}^{t}\{\varepsilon(\boldsymbol{x},t-\tau)\}^T\frac{d}{d\tau}[D(\tau)]d\tau\\&=\{\varepsilon(\boldsymbol{x},t)\}^T[D(0)]\\&\quad-\int_{0^+}^{t}\{\varepsilon(\boldsymbol{x},t)\}^T\frac{d}{d\tau}[D(t-\tau)]d\tau\end{aligned}\tag{30}$$

where the superscript T means the transpose of the matrix.

The strain in Eq. (13) can be written in matrix form as

$$\begin{aligned}\{\varepsilon(\boldsymbol{x},t)\}&=\sum_{n=1}^{N}\Phi_n(t)\{\varepsilon_{ij}^n(\boldsymbol{x})\}\\&=\sum_{n=1}^{N}\Phi_n(t)\{\varepsilon(\boldsymbol{x},n\Delta t)\}\end{aligned}\tag{31}$$

And substitution of Eq. (31) into Eq. (30) yields

$$\begin{aligned}\{\sigma(\boldsymbol{x},t)\}^T=&\{\varepsilon(\boldsymbol{x},t)\}^T[D(0)]-\sum_{n=1}^{N}\{\varepsilon(\boldsymbol{x},n\Delta t)\}^T\\&[D\{(N-n)\Delta t\}-D\{(N-n+1)\Delta t\}]\end{aligned}\tag{32}$$

Note that the summation operator in Eq. (32) is performed from $n=1$ to $n=N$ while the upper limit of that in Eq. (19) is $n=N-1$.

5. Numerical Examples

In the previous section, the relaxation modulus in shear is assumed to behave like a standard linear solid, which is illustrated by the spring-dashpot model made up of two springs (G_0, R) and one dashpot (η) as shown in Fig. 1. The stress-strain behaviour of that model could be described by either the hereditary integral as in Eq. (3) or the differential equation (Flügge, 1975) as

$$\sigma+p_1\dot{\sigma}=q_0\varepsilon+q_1\dot{\varepsilon}\tag{33}$$

where, $p_1=\eta/(G_0+R)$, $q_0=G_0R/(G_0+R)$, and $q_1=G_0\eta/(G_0+R)$.

So there exist some relations between the coefficients of Eqs. (23) and (33), which are given as

$$\begin{aligned}&G_0=q_1/p_1,\ \alpha=q_0p_1/q_1=R/(G_0+R)\\&\lambda=t_R^{-1}=p_1^{-1}=(G_0+R)/\eta\end{aligned}\tag{34}$$

And the viscoelastic material data for numerical examples are given as follows (Goudreau, 1970):

$$\rho=1.8\times10^{-5}\ \mathrm{Nsec^2/cm^4},\ G_0=1.275\times10^5\ \mathrm{N/cm^2}$$

$$K_0=2.35\times10^5\ \mathrm{N/cm^2},\ \alpha=0.098,\ \lambda=1\times10^6\ \mathrm{sec^{-1}}$$

$$\eta=0.1413\ \mathrm{Nsec/cm^2},\ R=1.386\times10^4\ \mathrm{N/cm^2}$$

From these data, the dilatational wave speed is calculated as $c_d=1.5\times10^5$ cm/sec.

5.1 Viscoelastic half-space subject to a sudden step pressure

Suppose that a half-space of the isotropic linear viscoelastic material is initially undisturbed and at $t=0$ its boundary is subject to a sudden step pressure $\sigma_0H(t)$. The finite element mesh with isoparametric quadratic quadrilateral elements ($L/l=24$, $l=0.0375$ cm) and boundary conditions are given as in Fig. 2, and the numerical analysis is performed under the condition of plane strain.

In Figs. 3 and 4, the present results for the wave propagation of stress obtained by using 24 finite elements are depicted at $t=2$ and 4 μsec and compared with Nickell's analytical solution (1968) and Goudreau's numerical results (1979) by the higher order mass model, and it is observed that the present results show a good agreement with Nickell's solution compared to Goudreau's results with spurious oscillations. We have also tried this computation using 12 finite elements and obtained the numerical results of almost the same accuracy except some more deviation near the wave front.

In Fig. 5, the effects of viscosity (η) on the

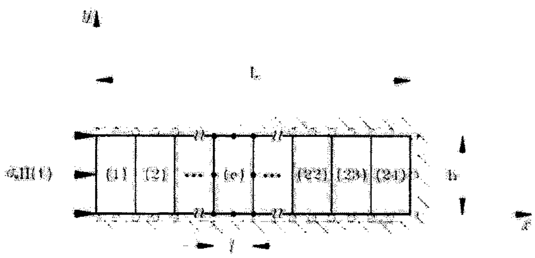

Fig. 2 Finite element model for the viscoelastic half-space

longitudinal displacements (u_x) are depicted at t=2, 4 and 6 μsec. It is observed that the displacement wave with a small viscocity (η=0.001413) follows behind the other two waves with higher viscosities due to the reduced stiffness of the viscoelastic material and reaches the equilibrium state in a short time, and the displacement becomes soon larger than the other two responses. Note that the slope of the curve, i.e., the strain, is constant when the equilibrium state is reached. The opposite response is obvious when the magnitude of viscosity becomes higher. For reference, the time step in this computation is $\Delta t=1.5625\times 10^{-8}$ sec, which is 1/16 times of the time for the viscoelastic compression wave to travel across a finite element.

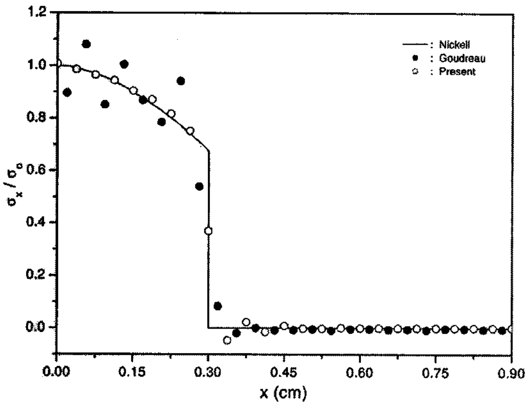

Fig. 3 Axial stress of the viscoelastic half-space at t=2 μsec

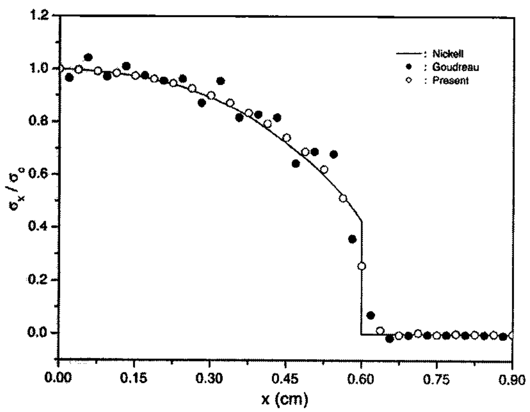

Fig. 4 Axial stress of the viscoelastic haif-space at t=4 μsec

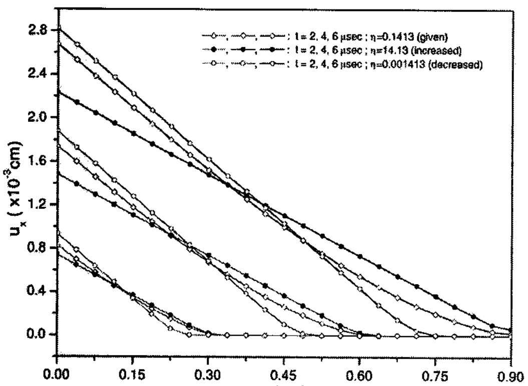

Fig. 5 Effects of the viscosity on the displacement wave propagation

5.2 Viscoelastic plate with a circular hole subject to a sudden tensile load

The viscoelastic plate (40×20×1 cm) with a circular hole (r=5 cm) is subject to a sudden tensile load $\sigma_0 H(t)$ at both ends. Capitalizing on the symmetry of the problem, a quarter of the plate is discretized as shown in Fig. 6, where the finite element mesh is composed of 76 elements and 269 nodes, and the numerical analysis is performed under the condition of plane stress. The displacement in the direction of y-axis and the normalized normal stress, i.e., the dynamic stress concentration factor, in the direction of x-axis at point A around the circular hole are depicted in Figs. 7 and 8, and it is observed from those two figures that as the value of α becomes smaller the curve of the viscoelastodynamic wave is shifted to the right with some dispersion compared to the elastodynamic solu-

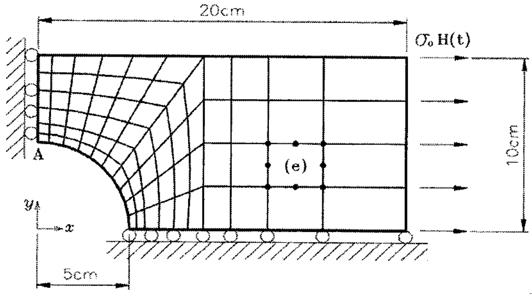

Fig. 6 Finite element model for the viscoelastic plate with a circular hole

tion ($\nu=0.2703$). It is hard to find the viscoelastodynamic solutions, whether numerical or theoretical, in the literature, so that the numerical data obtained by running the elastodynamic program (Sim and Lee, 2002) which has been tested by various ways are adopted for the comparison of the numerical results. For reference, The time step for this computation is $\Delta t = 2.691 \times 10^{-6}$ sec, and $\nu=0.2703$ is the Poisson's ratio corresponding to the initial relaxation moduli G_0 and K_0. In Fig. 7, the displacement curve for the case of $\alpha=0.098$ is omitted because it shows a relatively too large curve with a single peak ($u_y=-1.0129$ cm at $t=0.00134$ sec) compared with the other curves.

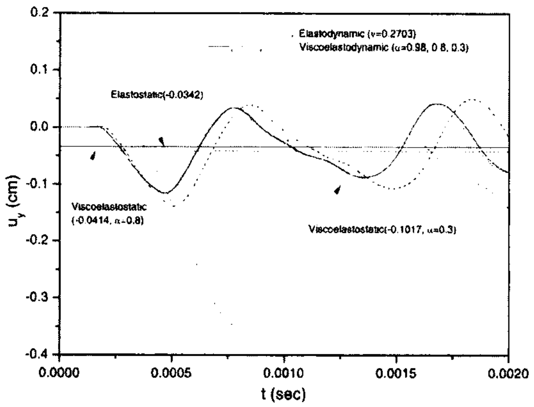

Fig. 7 Vertical displacements at point A on a circular hole in the viscoelastic plate

5.3 Wave diffraction by a cylindrical cavity in an infinite viscoelastic medium

Consider a long cylindrical cavity in an infinite isotropic linear viscoelastic medium impinged upon by a compressional P-wave whose front is parallel to the axis of cavity. Manolis and Beskos (1981) solved this problem in the Laplace domain by using the BEM only for the Maxwell and Kelvin materials.

In Fig. 9, the finite element model is given for this analysis, where only a half of an infinite domain is discretized by 238 finite elements utilizing the symmetry of the problem about the x-axis, and the artificial boundary is constructed far away from the cylindrical cavity to avoid the undesirable reflection.

In Figs. 10 and 11, the x-displacements and dynamic stress concentration factors at the point of $\theta=90°$ on the boundary of the cylindrical cavity are computed under the condition of plane strain and compared with the elastodynamic numerical results (Sim and Lee, 2002) for the variation of α values (0.098, 0.5, 0.8). It is observed from those two figures that the viscoelastodynamic curve adheres closely to the elastodynamic curve as the value of α approaches to unity, i.e., the damping of the material decreases, and the viscoelastic wave speed becomes lower due to the relaxed stiffness of the viscoelastic material and the attenuation increases as the value of α approaches to zero. For reference when the value

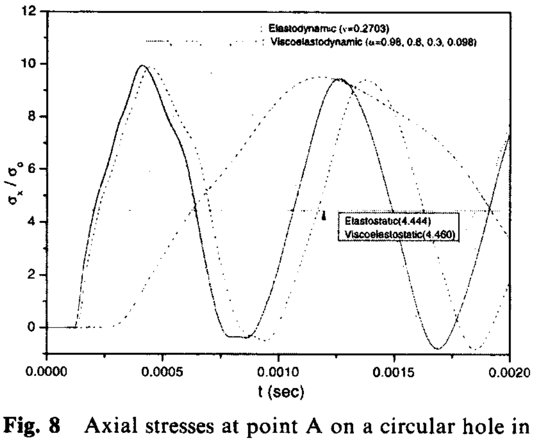

Fig. 8 Axial stresses at point A on a circular hole in the viscoelastic plate

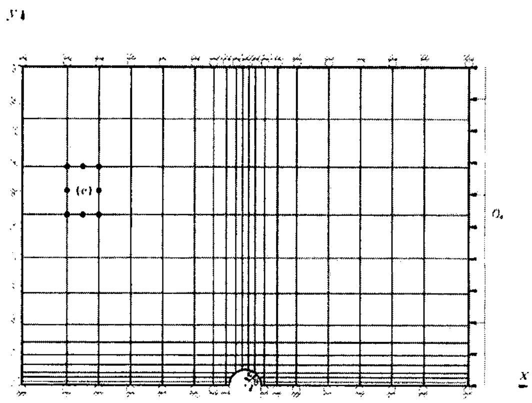

Fig. 9 Finite element model for the infinite viscoelastic medium with a cylindrical cavity

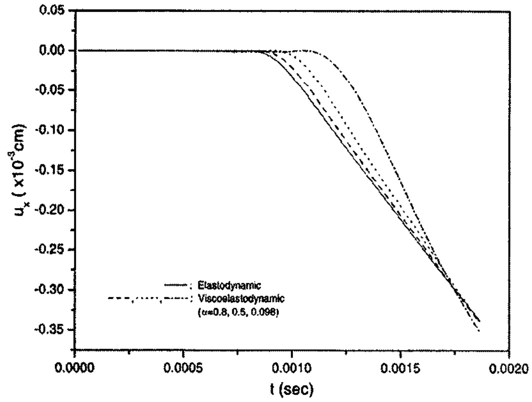

Fig. 10 Axial displacements at $\theta=90°$ on a cylindrical cavity in the infinite viscoelastic medium

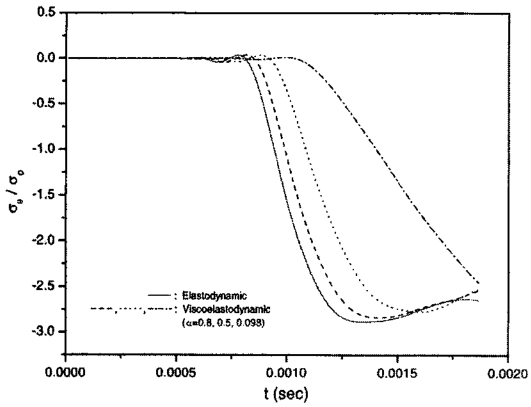

Fig. 11 Tangential stresses at $\theta=90°$ on a cylindrical cavity in the infinite viscoelastic medium

of α is equal to zero the standard linear solid model becomes the Maxwell model. The maximum values of the dynamic stress concentration factors for $\alpha=0.8$, 0.5, and 0.098 are -2.836 ($t=1.43\times10^{-3}$ sec), -2.772 ($t=1.60\times10^{-3}$ sec), and -2.456 ($t=1.87\times10^{-3}$ sec), respectively, which are compared with -2.884 ($t=1.33\times10^{-3}$ sec) and -2.67, in the case of the dynamic and static elasticities, respectively. For reference, the time step for this computation is $\Delta t=1.3333\times10^{-5}$ sec.

6. Conclusion

Time-domain finite element method based on the equations of motion in time integral has been presented for the general analysis of the transient dynamic linear viscoelastic problems. The weak form is obtained by applying the Galerkin's method to those equations and discretized spatially to obtain the semidiscrete equations in time integral. In the temporal approximation, the stepwise time interpolation functions are adopted to approximate the dependent variables on the discretized time axis, the simplified and unconditionally stable finite element equations being obtained. Two-dimentional examples with infinite and finite viscoelastic mediums have been solved successfully for the dynamic stress concentrations by the wave diffraction, which solutions are given nowhere in the literature. Currently, we are applying this method to the viscoelastodynamic fracture positively. So it may be said that this method is one of the useful numerical tools for the transient dynamic viscoelastic analysis.

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Appendix

The coefficients of the viscoelasticity matrix

$m_1=12K_0G_0+4G_0^2$

$m_2=12K_0G_0\lambda+12K_0G_0\alpha\lambda+8G_0^2\alpha\lambda$

$m_3=12K_0G_0\alpha\lambda^2+4G_0^2\alpha^2\lambda^2$, $m_4=3K_0G_0+4G_0^2$

$m_5=3K_0G_0\lambda+3K_0G_0\alpha\lambda+8G_0^2\alpha\lambda$

$m_6=3K_0G_0\alpha\lambda^2+4G_0^2\alpha^2\lambda^2$

$n_1=6K_0G_0-4G_0^2$

$n_2=6K_0G_0\lambda+6K_0G_0\alpha\lambda-8G_0^2\alpha\lambda$

$n_3=6K_0G_0\alpha\lambda^2-4G_0^2\alpha^2\lambda^2$, $n_4=3K_0G_0-2G_0^2$

$n_5=3K_0G_0\lambda+3K_0G_0\alpha\lambda-4G_0^2\alpha\lambda$

$n_6=3K_0G_0\alpha\lambda^2-2G_0^2\alpha^2\lambda^2$

$a_1=3K_0+4G_0$, $a_2=3G_0$

$b_1=6K_0\lambda+4G_0\alpha\lambda+4G_0\lambda$, $b_2=3G_0\lambda+3G_0\alpha\lambda$

$c_1=3K_0\lambda^2+4G_0\alpha\lambda^2$, $c_2=3G_0\alpha\lambda^2$

$$\gamma_1,\ \gamma_2=-\frac{-b_1\pm\sqrt{b_1^2-4a_1c_1}}{2a_1}$$

$$\xi_1,\ \xi_2=-\frac{-b_2\pm\sqrt{b_2^2-4a_2c_2}}{2a_2}$$

$$p_1=\frac{m_3}{a_1\gamma_1\gamma_2},\quad p_2=\frac{m_1\gamma_1^2-m_2\gamma_1+m_3}{\gamma_1a_1(\gamma_1-\gamma_2)}$$

$$p_3=\frac{m_1\gamma_2^2-m_2\gamma_2+m_3}{\gamma_2a_1(\gamma_2-\gamma_1)},\quad p_4=\frac{m_6}{a_2\xi_1\xi_2}$$

$$p_5=\frac{m_4\xi_1^2-m_5\xi_1+m_6}{\xi_1a_2(\xi_1-\xi_2)},\quad p_6=\frac{m_4\xi_2^2-m_5\xi_2+m_6}{\xi_2a_2(\xi_2-\xi_1)}$$

$$q_1=\frac{n_3}{a_1\gamma_1\gamma_2},\quad q_2=\frac{n_1\gamma_1^2-n_2\gamma_1+n_3}{\gamma_1a_1(\gamma_1-\gamma_2)}$$

$$q_3=\frac{n_1\gamma_2^2-n_2\gamma_2+n_3}{\gamma_2a_1(\gamma_2-\gamma_1)},\quad q_4=\frac{n_6}{a_2\xi_1\xi_2}$$

$$q_5=\frac{n_4\xi_1^2-n_5\xi_1+n_6}{\xi_1a_2(\xi_1-\xi_2)},\quad q_6=\frac{n_4\xi_2^2-n_5\xi_2+n_6}{\xi_2a_2(\xi_2-\xi_1)}$$

$$w_1=G_0\alpha,\quad w_2=\frac{G_0\lambda-G_0\alpha\lambda}{\lambda}$$